# Fundamentals of Zero-Order

## Binary Truth-Functional (Classical)

## and Modal Monotonic Logics

**(pretentious name)**

**(collected notes)**

**(excerpt and incomplete)**

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**INTRODUCTION**

**[ℑ 0.0]** In general, “logic is the formal science of truth”[[1]](#footnote-1) – how truth-values can be derived from other truth-values using various inference rules and deduction.

**[ℑ 0.1]** A *zero-order* logic deals in propositions or declarative sentences which can be assigned a truth value (in our present case, we will be considering binary truth valued systems only).

**[ℑ 0.2]** A *first-order* logic is an extension of the zero-order calculus. To the zero-order calculus we add a set of quantifiers {∃, ∀}, additional rules of inference, object and relation variables, as well as supplemented wff formation rules.

**[ℑ 0.3]** Briefly, a “language is determined by its symbols along with its syntactic formation rules. A calculus [alternatively called an axiomatic system] is a language together with axioms and/or rules of inference for making deductions within the formal language. A logic is a language along with a semantics to interpret that language.”[[2]](#footnote-2)

# ZERO-ORDER BINARY TRUTH-FUNCTIONAL (CLASSICAL) LOGIC

**[ℑ 0.4]** For our purposes we will employ a simple propositional calculus of the axiomatic form developed by Łukasiewicz.[[3]](#footnote-3)

**[ℑ 1.0]** We take our *propositional calculus* (alternatively called a *sentential calculus*) to be called L1. We define L1 as per the following:

**[ℑ 1.1]** L1 = {*A*, *Z*, *I*, Ω}

**[ℑ 1.2]** *A* is a (presumably) finite set of *propositional variables* (alternatively: *sentence letters*), such that *A* = {A00, A10, …, B00, B10, …, …, Z00, Z10, …}

**[ℑ 1.3]** Ω is the set of primitive *logical operators* (*logical connectives*) for L1, such that:

[a] Ω = Ω0 ∪ Ω1 ∪ Ω2

[b] Where Ω0 is the set of logical connectives of *arity* 0,

such that Ω0 = {⊥, Τ}

[c] Where Ω1 is the set of logical connectives of *arity* 1,

such that Ω1 = {¬}

[d] Where Ω2 is the set of logical connectives of *arity* 2,

such that Ω2 = {→}

**[ℑ 1.4]** The *well-formed formulae* (wff) of L1 are recursively defined as follows:

[a] Any δ, where δ is a propositional variable of L1, is a formula.

[b] If δ is a formula then, ¬δ is a formula.

[c] If δ and ϕ[[4]](#footnote-4) are formulas then, δ → ϕ is a formula.

[d] T and ⊥ are formulas.

[e] There are no other wff.

**[ℑ 1.5]** *Z* is the zeta set of *inference rules* valid in L1. We will here include only the *modus ponens*, such that *Z* = {*p*, *p* → *q* ├ *q*}

**[ℑ 1.6]** *I* is the set of *axiom schemata* for our logic L1, such that:

[a] *I* = AS1 ∪ AS2 ∪ AS3

[b] AS1 = {A → (B → A)}

[c] AS2 = {(A → (B → C)) → ((A → B) → (A → C))}

[d] AS3 = {(¬A → ¬B) → (B → A)}

**[ℑ 1.7]** It should suffice for now to stipulate that each *axiom schemata* of AS(V) is a tautology.[[5]](#footnote-5) An *axiom schema* is a wff in the language of an axiomatic system, in which one or more schematic variables (which are metalinguistic constructs) appear and which stand for any wff of the system.

**[ℑ 1.8]** We define a *theorem* to be a statement proved from the application of our inference rules and *axiom schemata* alone, without any additional *premises* (assumptions).

**[ℑ 1.9]** We follow standard convention regarding parenthetical dropping, precedence, quotation and uniform substitution.

**[ℑ 2.0]** We establish the following logical equivalences (from which we can define the remaining standard logical connectives):

[a]A → ⊥ ≡ ¬A

[b] T → A ≡ A

[c] A → B ≡ ¬(A ∧ ¬B)

[d] A ∧ B ≡ ¬(¬A ∨ ¬B) ≡ ¬(A → ¬B)

[e] A ∨ B ≡ ¬A → B

[f] A ↔ B ≡ ¬((A→ B) → ¬(B → A)) ≡ (A → B) ∧ (B → A)

### ZERO-ORDER MODEL THEORY

**[ℑ 3.0]** Let *V* be a theory (a set of *wff*) such that:

[a] *V* = *atom*(*V*)∪ *complex*(*V*)

[b] Where atom(*V*) is the set of propositional variables (alternatively called *atomic formulas*) in *V*.

[c] Where complex(*V*) is the set of non-atomic formulas in *V*.

[d] Where each member *a* ∈ *atom*(*V*) is a *wff* in *L*1.

[e] Where each member *b* ∈ *complex*(*V*) is a *wff* in *L*1.

**[ℑ 3.1]** A *zero-order* structure is a set *S* = 〈*V*, Φ, Φ\*〉 such that:

[a] Φ : *atom*(*V*) → {*T*, *F*} such that:

[b] For each *a* ∈ *atom*(*V*): Φ assigns a *truth-function* Ψ such that:

[c] Ψ(*a*) = *T* else Ψ(*a*) = ⊥

[d] We call Φ an *interpretation* or an assignment of *truth-values* to

atomic formulas of *L*1.

[e] Φ\*: *V* → {*T*, *F*} such that:

[f] For any *p, q* ∈ *V*: Φ\* assigns a truth-function Ψ\* such that:

[g] Ψ\*(*p*) = Ψ(*p*)

[h] Ψ\*(⊥) = ⊥

[i] Ψ\*(*T*) = *T*

[j] Ψ\*(¬*p*) = *T* *iff* Ψ(*p*) = ⊥

[k] Ψ\*(*p* → *q*) = *T* *iff* Ψ(*p*) = ⊥ or Ψ(*q*) = *T*

[l] Ψ\*(*p* & *q*) = *T* *iff* Ψ(*p*) = *T* and Ψ(*q*) = *T*

[m] Ψ\*(*p* ∨ *q*) = *T* *iff* Ψ(*p*) = *T* or Ψ(*q*) = *T*

[n] Ψ\*(*p* ↔ *q*) = *T* *iff* Ψ(*p*) = Ψ(*q*)

[\*] If a theory *T* has *n* propositional variables then, under an *m*-valued truth-functional logic there are *m*n possible interpretations for T.

**[ℑ 3.2]** If every sentence *a* ∈ *V* is true (alternatively, if every sentence *a* ∈ *V* is given a truth-value *T*) under a structure *S*, then we say that *M* is a model of *V* (*M* satisfies *V*).

**[ℑ 3.3]** A sentence *a* that is true under every interpretation (alternatively, a sentence *a* that is true by every structure) is called a *logical truth* (or as some say *valid*).

[a] All *tautologies* are logical truths.

**[ℑ 3.4]** A theory *V* is *consistent* if there exists at least one structure that is a model of *V*.

### AXIOM SYSTEMS

**[ℑ 4.0]** An *axiom system* *S* is a set of axioms *I* and a set of inference rules *Z*.

**[ℑ 4.1]** An axiom system *S* is *sound* *iff* each sentence *s* that is provable in system *S* is true.

[a] If axiom system *S* has only tautologies as axioms and has *modus ponens* as its only rule of inference then, axiom system *S* is *sound*.

**[ℑ 4.2]** An axiom system *S* is *complete* *iff* each sentence *s* that is true is provable in system *S*.

[a] By proving that a complete system *M* can be proven in *S*, one can show that *S* is also complete.

**[ℑ 4.3]** An inference rule ⊢is *sound* *if P* ⊢ *Q* implies *P* ⊨ *Q*.

**[ℑ 4.4]** An inference rule ⊢is *complete* if *P* ⊨ *Q* implies *P* ⊢ *Q*.

### FIRST-ORDER (PREDICATE) BINARY TRUTH-FUNCTIONAL LOGIC

**[ℑ 5.0]** We now expand on *L*1 by adding the basic apparatus of a first-order[[6]](#footnote-6) calculus. We will call our first-order calculus *L*2. Simply, we can define *L*2 by adding the following to *L*1: *variable*, *quantifier, relation*, and *function* (*operation*)symbols along with additional axiom schemata, rules of inference and *wff* formation rules.

[\*] For brevity’s sake, we will only treat some of these basic notions here:

**[ℑ 5.1]** *Arity* is the number of arguments that a relation or a function can take.

**[ℑ 5.2]** The *relation*[[7]](#footnote-7) symbols in the *alphabet* (*vocabulary*) of L2 comprise a set of n-ary relation symbols usually denoted as per the following:

[a] {*R*00, *R*10, …, *R*01, *R*11, …, *R*02, *R*12, …, …, *R*0n, *R*1n,…}

[b] A *predicate* is an unary relation.

[c] A sentence letter is a zero-place (zero-arity) relation symbol.

[d] The *function*[[8]](#footnote-8) symbols in the alphabet of L2 comprise a set of n-ary function symbols usually denoted as per the following:

{ƒ00, ƒ10, …, ƒ01, ƒ11, …, ƒ02, ƒ02, …, …, ƒ0n, ƒ1n,…}

[e] Where a *constant* symbol is a zero-place (zero-arity) function symbol.

**[ℑ 5.3]** The *variable*[[9]](#footnote-9) symbols in the alphabet of L2 comprise a set of variable symbols usually denoted as per the following: {n0, n1, …, o0, o1, …, …, z0, z1, …}

**[ℑ 5.4]** The set of *quantifier* symbols in the alphabet of L2 is as follows: {∃}[[10]](#footnote-10)

**[ℑ 5.5]** We add two axiom schemata to the set of axiom schemata in L1:

[a] AS4 = {∀x *F*(x) → *F*(y)}

[b] AS5 = {*F*(y) → ∃x *F*(x)}

[c] Where *F*(x) is any sentential formula in which x occurs free, y is a term, *F*(y) is the result of substituting y for the free occurrences of x in sentential formula *F*, and all occurrences of all variables in y are free in *F*.[[11]](#footnote-11)

**[ℑ 5.6]** We join *modus ponens* with two additional rules of inference.

[a] *Z* is the zeta set of inference rules valid in L2, such that:

[b] *Z* = MP ∪ I1 ∪ I2

[c] MP = {p, p → q ├ q}

[d] I1 = {*G* → *F*(x) ├ *G* → ∀x *F*(x)}

[e] I2 = {*F*(x) → *G* ├ ∃x *F*(x) → *G*}

[f] Where *F*(x) is any sentential formula in which x occurs as a free variable and x does not occur as a free variable in formula *G*.[[12]](#footnote-12)

**[ℑ 5.7]** We follow standard convention regarding parenthetical dropping, wff and term formation, quotation and uniform substitution.

**FIRST-ORDER MODEL THEORY**[[13]](#footnote-13)

**[ℑ 6.0]** A first-order *structure* *K* is defined as an ordered pair {|*K*|, Φ}.

[a] |*K*| is the *universe* or *domain of discourse* for the *structure* *K*.

[b] Φ is a function whose domain is a set of non-logical symbols.

[c] This domain is called the *signature* of *K*, such that:

[d] sig(*K*) = {*R*1, *R*2, *R*3, …, *R*m, ƒ1, ƒ2, ƒ3, …, ƒz, *c*1, *c*2, *c*3, …, *c*w}

[e] To each *n*-ary relation symbol *R* ∈ sig(*K*) we assume that Φ assigns an *n*-ary relation: *R* ⊆ |*K*|n

[f] To each *n*-ary *function* symbol ƒ ∈ sig(*K*) we assume that Φ assigns an *n*-ary function, such that ƒ: |*K*|n → |*K*|

[g] To each constant symbol *c* ∈ sig(*K*) we assume that Φ assigns an individual constant: *c* ∈ |*K*|

**[ℑ 6.1]** Given a structure *K* and a *sentence* σ such that sig(σ) ⊆ sig(*K*) we write ‘*K*╞ σ’ or K *satisfies* σ (σ is *true* in K).

**[ℑ 6.2]**  Let *S* be a set of sentences. A *model* of *S* is a structure *M* such that

*M*╞ σ for all σ ∈ *S*, and sig(*M*) = sig(*S*).

**[ℑ 6.3]**  The *class of all models* of *S* is denoted Mod(*S*). A sentence τis said to be a

*logical consequence* of *S* (written *S╞* τ) if sig(τ) ⊆sig(*S*), and *M╞* τ for

all *M* ∈ Mod(*S*).

**[ℑ 6.4]** A *theory* is a set *T* of sentences which is consistent and closed under logical consequence; in other words, *T* has at least one model, and τ∈ *T* whenever τis a sentence such that sig(τ) ⊆sig(*T*) and *M╞* τ for all *M* ∈ Mod(*S*).

**[ℑ 6.5]** If *A* is a model class, we write Th(*A*) for the *theory of A*, i.e. the set of

sentences σ such that sig(σ) ⊆sig(*A*) and *M* ╞ σ for all *M* ∈ *A*.

**[ℑ 6.6]** If *T* is a theory and *S* ⊆ *T*, we say that *S* is a *set of axioms* for *T* if

*T* = Th(Mod(*S*)). If there exists a finite set of axioms for *T*, we say that *T*

is *finitely axiomatizable*.

**[ℑ 6.7]** Two structures *K* and *B* are said to be *isomorphic* (written *K* ≅ *B*) if

sig(*K*) = sig(*B*) and there exists an isomorphic map of *K* onto *B*, i.e.

*i* : |*K*| → |*B*| such that:

[a] *i* is one-one and onto;

[b] *RK*(*a*1, …, *an*) if and only if *RB*(*a*1, …, *an*);

[c] *i*(*ƒK*(*a*1, …, *an*)) = *ƒB*(*i*(*a*1), …, *i*(*an*));

[d] *i*(*cK*) = *cB*.

### ZERO-ORDER BINARY TRUTH-FUNCTIONAL MODAL LOGIC

**[ℑ 7.0]** We now expand on L1 by adding to its basic apparatus additional logical operators, a different set of recursively-defined wff, and supplementary axiom schemata. We will call our resulting modal calculus L3.

**[ℑ 7.1]** We add to the unary *logical operators* Ω1 in L1 we add the following: {□}[[14]](#footnote-14).

**[ℑ 7.2]**  The *well-formed formulae* (wff) of L3 are recursively defined as follows:

[a] Any δ, where δ is a propositional variable of L, is a formula.

[b] If δ is a formula then, ¬δ is a formula.

[c] If δ and ϕ are formulas then, δ → ϕ is a formula.

[d] T and ⊥ are formulas.

[e] If δ is a formula then, □δ is a formula.

[f] There are no other wff.

**[ℑ 7.3]**  We add two axiom schemata to the set of axiom schemata in L1:

[a] **N** If p is a theorem, then □p is a theorem

[b] **K** □(p → q) → (□p → □q)

**[ℑ 7.4]** There are different axiom schemata that are regularly added **N**, **K** above:

[a] **D** (□p) → (◊p)

[b] **T** (□p) → p

[c] **B** p → (□◊p)

[d] **S4** (□p) → (□□p)

[e] **S5** (◊p) → (□◊p)

**[ℑ 7.5]** We obtain the following modal axiom systems upon adding the above axiom schemata to L3:

[a] System *K* =df **N** + **K**

[b] System *T* =df System *K* + **T**

[c] System *S4* =df System *T* **+ S4**

[d] System *S5* =df System *S4* + **B** (alternatively: **T** + **S5**)

[e] System *D* =df System *K* + **D**

### ZERO-ORDER MODAL MODEL THEORY[[15]](#footnote-15)

### [ℑ 8.0] A *relational structure* (also called a possible worlds model, Kripke model or a modal model) is a triple *M* = {*W, R, V*} such that:

### [a] *W* is a nonempty set (elements *w* of *W* are called states),

### [b] *R* is a relation on *W* (formally, *R* ⊆ *W*× *W*) and,

### [c] *V* is a valuation function assigning truth values *V*(*p, w*) to atomic propositions *p* at state *w* (formally *V* : *A* × *W* → {⊥, T} where *A* is the set of sentence letters).

**[ℑ 8.1]** Truth of a modal formula *p* at a state *w* in a relational structure *M* = {*W, R, V*} is denoted *M,w╞* *p* and isinductively defined as follows:

[a] *M,w╞* *p*  *iff* *V*(*p, w)* = T (where *p* ∈ *S*)

[b] M,w╞ T and *¬*(M,w╞ ⊥)

[c] *M,w╞ ¬p iff* *¬*(*M,w╞ p*)

[d] *M,w╞ p & q iff M,w╞ p & M,w╞ q (where p,q* ∈ *S)*

[e] *M,w╞* □*p iff* (∀ *v* ∈ *W*)(*wRv →* *M,v╞* *p*)

[f] *M,w╞* ◊*p iff (*∃ *v* ∈ *W*)(*wRv* & *M,v╞* *p*)

**BIBLIOGRAPHY AND WORKS CITED**

Doets, Kees. Basic Model Theory. Stanford: CSLI Publications, 1996.

Hodges, Wilfrid. “Model Theory.” Stanford Encyclopedia of Philosophy. 20

July 2009. 29 November 2009. <http://plato.stanford.edu/entries/model-theory/>

Mattison, Robert. An Introduction to the Model Theory of First-Order Predicate Logic and a Related Temporal Logic. California: Rand Corporation, 1969.

Ó Dúnlaing, Colm. “Completeness of some axioms of Lukasiewicz's: an exercise in problem-solving.” Trinity College Dublin School of Mathematics. 1997. 27 October 2010. <ftp://ftp.maths.tcd.ie/pub/tcdmath/tcdm9705.ps.Z>

Pacuit, Eric. “Notes on Modal Logic.” Stanford University Philosophy Department. 25 January 2009. 27 October 2010. <http://www.jakubszymanik.com/ML/ml-notes.pdf>

Sakharov, Alex. “First-Order Logic.” Wolfram Math. 11 October 2010. 27

October 2010. <http://sakharov.net/foundation.html>

Simpson, Stephen G. “Math 563: Model Theory.” Pennsylvania State Department of Mathematics. 2 May 1998. 4 April 2009. <www.math.psu.edu/simpson/courses/math563/>

Smith, Leslie. “Piaget’s Model” in Blackwell Handbook of Childhood Cognitive Development. Ed. Usha Goswami. pp. 515-538. Blackwell Publishing: UK, 2004.

1. as stated by Frege, see (Smith, 527) [↑](#footnote-ref-1)
2. his emphasis; (Mattison, 1) [↑](#footnote-ref-2)
3. see (Ó Dúnlaing) [↑](#footnote-ref-3)
4. δ and ϕ are *meta-variables* which range over the expressions of L1. The expressions of L1 form the set of all possible concatenations of *propositional variables* (alternatively: *sentence letters*) and/or logical connectives of L1. [↑](#footnote-ref-4)
5. A proof via truth-table is provided in the appendix [↑](#footnote-ref-5)
6. A logic which deals with a single-sorted domain of discourse and whose quantifiers do not range over sets, relations or predicates. [↑](#footnote-ref-6)
7. A relation is an operator which assigns elements in the domain of discourse to a truth value. [↑](#footnote-ref-7)
8. Functions are those operators which assign elements in the domain of discourse to other elements in the domain of discourse. [↑](#footnote-ref-8)
9. These variables range over the domain of discourse. [↑](#footnote-ref-9)
10. We define ∀ as per the following : (∀x A :df ¬∃x¬A) [↑](#footnote-ref-10)
11. We here use the axiom schemata outlined in (Sakharov). [↑](#footnote-ref-11)
12. (Ibid.) [↑](#footnote-ref-12)
13. Definitions from (Simpson, 9-11) [↑](#footnote-ref-13)
14. From □ we define ◊ : □p ≡ ¬◊¬p [↑](#footnote-ref-14)
15. see (Pacuit) [↑](#footnote-ref-15)